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Wiener index of bridge connected maximum planar graphs

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Abstract: Let G_1, G_2, ...G_n be n distinct maximum planar isomorphic graphs. $n - 1$ edges are connected between all the n graphs whose end vertex are some G_i and G_j, except the end graphs to form a connected graph G defined as *n-bridge-connected* is introduced. This paper focus on finding bridge connected specifically for n star graphs and cycle graphs.

Keywords: Wiener Index, Planar Graph, Cycle Graph, star graph, neural network.

1. Introduction

A *Graph* denoted by $G = G\,(V\,(G), E(G))$, where $V\,(G)$ is the vertex set and $E(G)$ is the edge set of G. The graphs considered in this paper are undirected, finite and simple. A graph G is said to be *connected* if there is a path between any two distinct vertices of G. Let $x, y \in V\,(G)$. Let $d\,(x, y)$ be the length of the shortest path from x to y. The distance between two vertices in a connected graph G is the number of edges in the shortest path between them. [8]

A *walk* of a graph G is a finite, alternating sequences of vertices and edges say, v_0, e_1, v_1, e_2, v_2, ..., v_{n-1}, e_n, v_n beginning with v_0 and ending with v_n such that each edge e_i is incident with v_{i-1} and v_i. The number n is called the length of the walk. An open walk in which no vertex appears more than once is called a *path*. A closed path is called a *cycle*. A graph G is called *acyclic* if it has no cycles. A connected acyclic graph is called a *tree*. One of the graph invariants is the topological index introduced by Harold Wiener in 1947, have significant attention in the field of Chemical Graph theory till date. This field has wide applications in Chemical sciences, medical sciences which in turn used as a tool for modelling chemical properties of molecular bonds and thereby able to study the structure of organic compounds.[10]

The *Wiener index* of a connected graph denoted by $W\,(G)$ is defined as the sum of all distances between every pair of vertices of G [15]. The Wiener index of a graph was the first reported topological index based on graph distances. Wiener (1947, 1948) was perhaps the first one to analyze some aspects of branching by fitting experimental data for several properties of alkane compounds, using the deviation of his path number W in branched alkanes from that of the linear isomeric compound [10].

$$W(G) = \frac{1}{2} \sum_{\substack{\{u_i, u_j\} \subseteq V(G) \\ i<j}}^{n} d(u_i, u_j) = \sum_{\substack{\{u_i u_j\} \subseteq V(G) \\ u_i < u_j}}^{n} d(u_i, u_j)$$

2. Preliminaries

A graph is said to be *planar*, if its edges intersect only at their end points. A simple graph G is said to be *Maximum planar*, if it is planar and adding any edge on the existing vertex dissatisfy planar property or in other words, there exist zero crossing of the edges in the graph G.

The maximum Wiener index of maximum planar graph has been widely studied.[9] Since maximum planar graphs resembles the structure of molecular bonding, the study connecting many maximum planar graphs into a single structure and finding the Wiener index helps to analyse the properties of molecules with n-distances such as, alkanes whose main commercial sources are petroleum and natural gas. [1 to 7]

The present study introduces *bridge- connected* in finding Wiener index for n-maximum planar graphs connected together to form a massive large molecular structure specifically for cycle and star graphs.

Cycle graphs C_n helps in studying cycloalkanes such as cyclobutene resembles C_4, Cyclopentane and methyl cyclopentane resembles C_5, Cyclohexane resembles C_6 and many other structures [9,10,13]. Star graphs resembles neurological structure in brain networks, wherein many star graphs are connected and finding the Wiener index of connected star graphs S_n helps to analyse assortativity index [5].

3. The Wiener index of n-bridge connected maximum planar graphs

Definition 4.1:

Bridge-connected: Two isomorphic maximum planar graphs are said to be 1-bridge-connected, if there exist an edge whose end points are connected to any vertex of two graphs. For the n-bridge-connected between maximum planar n-graphs, $G_1, G_2, ..., G_n$, there exist $n-1$ edges connected, such that there exists unique edge between any G_i and G_j except between G_1 and G_n, symmetrically arranged in particular order to form a single connected graph G, such that even after rotation of these graphs retains the original structure.

Basically, most of the molecules appear in cycle manner, the cycle graph C_m, $m \geq 4$ are taken into account which is followed in this next section.

4. The Wiener index of n-bridge connected cycle graphs C_m, m is odd or even:

Example 4.1:

4.1.1. The Wiener index of 2-bridge connected C_4 graphs:

Let G_1 and G_2 be two non-overlapping C_4 graph with vertex sets $\{u_1, u_2, u_3, u_4\}$ and $\{v_1, v_2, v_3, v_4\}$ respectively. Two graphs $W(G_1) +_2 W(G_2)$ is obtained by joining any vertex of G_1 to any vertex of G_2 by new edge.

Since $W(C_4) = \frac{1}{8}(4)^3 = 8$

$$W(G_1) = \sum_{i<j}^{n} d(u_i, u_j) = 8, W(G_2) = \sum_{i<j}^{n} d(v_i, v_j) = 8$$

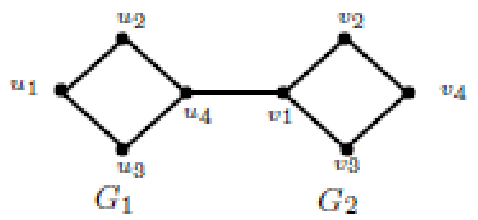

Figure.1: 2-bridge connected C_4 graphs

Since G_1 and G_2 are connected by an edge (u_4, v_1)

$$W(G_1) + W(G_2) = W(G_1) + W(G_2) +(W(G_1) +_2 W(G_2))$$

where, $W(G_1) +_2 W(G_2)$

$$= \sum_{i,j=1}^{4} d(v_i, u_j)$$

$= d(v_1, u_1) + ... + d(v_1, u_4)$
$+ d(v_2, u_1) + ... + d(v_2, u_4)$
$+d(v_3, u_1) + ... + d(v_3, u_4)$
$+ d(v_4, u_1) + ... + d(v_4, u_4)$
$= (1+2+2+3)+2(2+3+3+4) +(3+4+4+5)$
$= [(n-1)+2n +(n+1)]+2[n+2(n+1) +(n+ 2)]+[(n + 1)+2(n+2)+(n+3)]$

= C_4 (U_2), say, Therefore,
$W(G_1) + W(G_2) = 2W(C_4) + C_4(U_2)$ …(4.1)
Calculation: $W(G_1) +_2 W(G_2)$
$= 2(8) + 8 + 2(12) + 16 = 64$

4.1.2. The Wiener index of 3-bridge connected C_4 graphs:

Let G_1, G_2 and G_3 be three non-overlapping C_4 graph with the vertex sets $\{u_1, u_2, u_3, u_4\}$, $\{v_1, v_2, v_3, v_4\}$ and $\{w_1, w_2, w_3, w_4\}$ respectively.3-bridge-connected graphs $W(G_1) +_2 W(G_2) +_3 W(G_3)$ is obtained by joining single edge that has end point in $W(G_1) +_2 W(G_2)$ and G_3 horizontally.

Since $W(C_4) = \frac{1}{8}(4)^3 = 8$ there will be 3 copies of C_4 $W(G_i) = 8, i = 1,2,3$. Then,

$$W(G_3) = \sum_{i<j}^{n} d(w_i, w_j) = 8$$

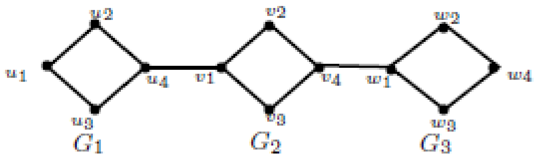

Figure.2: 3-bridge connected C_4 graphs

Since $W(G_1) +_2 W(G_2)$ and G_3 are connected by an edge (v_4, w_1)
$W(G_1) + W(G_2) + W(G_3) =$
$W(G_1) + W(G_2) + W(G_3)$
$+(W(G_1) +_2 W(G_2)) + (W(G_2) +_2 W(G_3))$
$+(W(G_1) +_2 W(G_2) +_3 W(G_3))$
where, $W(G_1) +_2 W(G_2) +_3 W(G_3)$

$$= \sum_{i,j=1}^{4} d(w_i, u_j)$$

$= d(w_1, u_1) + \ldots + d(w_1, u_4)$
$+ d(w_2, u_1) + \ldots + d(w_2, u_4)$
$+ d(w_3, u_1) + \ldots + d(w_3, u_4)$
$+ d(w_4, u_1) + \ldots + d(w_4, u_4)$
$=(4+5+5+6)+2(5+6+6+7)+(6+7+ +8)$
$= [(n + 1) + 2(n + 2) + (n + 3)]$
$+ 2[(n + 2) + 2(n + 3) + (n + 4)]$
$+[(n + 3) + 2(n + 4) + (n + 5)]$
$= C_4(U_3)$ (say). Therefore,
$W(G_1) + W(G_2) + W(G_3) =$
$3W(C_4) + 2C_4(U_2) + C_4(U_3)$ …..(4.2)
Calculation: $W(G_1) +_2 W(G_2) +_3 W(G_3)$
$= 3(8) + 2(48) + 20 + 2(24) + 28 = 216$

4.1.3. The Wiener index of 4-bridge connected C_4 graphs:

Let G_1, G_2, G_3 and G_4 be four non-overlapping C_4 graphs with the vertex sets $\{u_1, u_2, u_3, u_4\}$,$\{v_1, v_2, v_3, v_4\}$,$\{w_1, w_2, w_3, w_4\}$ and $\{x_1, x_2, x_3, x_4\}$ respec- tively. 4-bridge-connected graphs $W(G_1) +_2 W(G_2) +_3 W(G_3) +_4 W(G_4)$ is obtained by joining single edge that has end point in $W(G_1) +_2 W(G_2) +_3 W(G_3)$ and G_4 horizontally.

Since $W(C_4) = \frac{1}{8}(4)^3 = 8$ there will be 4 copies of C_4 , $W(G_i) = 8, i = 1,2,3,4$. Then,

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$$W(G_4) = \sum_{i<j}^{n} d(x_i, x_j) = 8$$

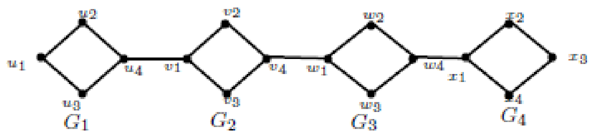


Figure.3: 4-bridge connected C_4 graphs

Since $W(G_1) +_2 W(G_2) +_3 W(G_3)$ and $W(G_4)$ are connected by an edge (w_4, x_1),
$W(G_1) + W(G_2) + W(G_3) + W(G_4)$
$= W(G_1) + W(G_2) + W(G_3) + W(G_4)$
$+(W(G_1) +_2 W(G_2)) + (W(G_2) +_2 W(G_3))$
$+ (W(G_3) +_2 W(G_4))$
$+ (W(G_1) +_2 W(G_2) +_3 W(G_3))$
$+ (W(G_2) +_2 W(G_3) +_3 W(G_4))$
$+ (W(G_1) +_2 W(G_2) +_3 W(G_3) +_4 W(G_4))$
where, $W(G_1) +_2 W(G_2) +_3 W(G_3) +_4 W(G_4)$

$$= \sum_{i,j=1}^{4} d(x_i, u_j)$$

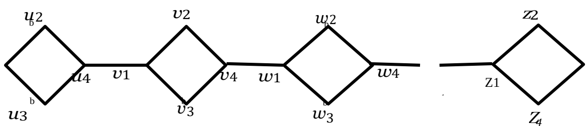
$= d(x_1, u_1) + \ldots + d(x_1, u_4)$
$+ d(x_2, u_1) + \ldots + d(x_2, u_4)$
$+d(x_3, u_1) + \ldots + d(x_3, u_4)$
$+ d(x_4, u_1) + \ldots + d(x_4, u_4)$
$=(7+8+8+9)+2(8+9+9+10)+(9+10+10+ 11)$
$= [(n + 3) + 2(n + 4) + (n + 5)]$
$+ 2[(n + 4) + 2(n + 5) + (n + 6)]$
$+[(n + 5) + 2(n + 6) + (n + 7)]$
$= C_4(U_4)$ (say). Therefore,
$W(G_1)+ \ldots + W(G_4) = 4W(C_4)$
$+3 C_4 (U_2)+ 2 C_4(U_3)+ C_4(U_4)$ …..(4.3)
Calculation:
$W(G_1) +_2 \ldots +_4 W(G_4) = 4(8) + 3(48) +$
$2(96) + 32 + 2(36) + 40 = 512$

4.1.4. The Wiener index of n-bridge connected C_4 graphs:

Let $G_1, G_2, \ldots, G_n$ be n non-overlapping C_4 cycle graphs with vertex sets $\{u_1, u_2, u_3, u_4\}$, $\{v_1, v_2, v_3, v_4\}, \ldots, \{z_1, z_2, z_3, z_4\}$ respectively. n-bridge-connected graphs $W(G_1) +_2 W(G_2) +_3 W(G_3) +_4 \ldots +_n W(G_n)$ is obtained by joining sin- gle edge that has end point in $W(G_1) +_2 W(G_2) +_3 \ldots +_{n-1} W(G_{n-1})$ and G_n horizontally. Therefore there exists a single edge whose end points lies in every pair of cycles $(G_1, G_2, \ldots G_n)$ (except the end cycles) which are bridge connected systematically arranged in horizontal manner.

Since $W(C_4) = \frac{1}{8}(4)^3 = 8$ there will be n copies of C_4 , $W(G_i) = 8, i = 1,2,3,4$. Then,

$$W(G_n) = \sum_{i<j}^{n} d(z_i, z_j) = 8$$



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U_1

G_1 G_2 G_3 G_n

Figure.4: n-bridge connected C_4 graphs

Since $W(G_1) +_2 W(G_2) +_3 \ldots +_{n-1} W(G_{n-1})$ and $W(G_n)$ are connected by an edge (y_4, z_1),

$W(G_1) + W(G_2) + \ldots + W(G_n)$
$= W(G_1) + W(G_2) + \ldots + W(G_n)$
$+(W(G_1) +_2 W(G_2)) + (W(G_2) +_2 W(G_3))$
$+ \ldots + (W(G_{n-1}) +_2 W(G_n))$
$+(W(G_1) +_2 W(G_2) +_3 W(G_3))$
$+ \ldots + (W(G_{n-2}) +_2 W(G_{n-1}) +_3 W(G_n))$
$+\ldots + (W(G_1) +_2 W(G_2) +_3 \ldots +_n W(G_n))$

where, $W(G_1) +_2 W(G_2) +_3 \ldots +_n W(G_n)$

$$= \sum_{i,j=1}^{4} d(z_i, u_j)$$

$= d(z_1, u_1) + \ldots + d(z_1, u_4)$
$+ d(z_2, u_1) + \ldots + d(z_2, u_4)$
$+ d(z_3, u_1) + \ldots + d(z_3, u_4)$
$+ d(z_4, u_1) + \ldots + d(z_4, u_4)$
$=1[(n - 1 + 2t) + 2(n + 2t) + (n + 1 + 2t)]$
$= + 2[(n + 2t) + 2(n + 1 + 2t) + (n + 2 + 2t)]$
$= + 1[(n + 1 + 2t) + 2(n + 2 + 2t)$
$+ (n + 3 + 2t)] = C_4(U_n)(say)$. Therefore,

$W(G_1)+ \ldots + W(G_n) = n\, W(C_4)$
$+(n-1)\, C_4(U_2)+ (n-2)\, C_4(U_3)+(n-3)\, C_4(U_4) + \ldots +1\, C_4(U_n)$.....(4.4)

4.1.5. **Theorem 4.1:** *Let $G_1, G_2, \ldots, G_n$ be n distinct non-overlapping cycle graphs, C_m, $m \geq 3$ with vertex sets $\{u_1, u_2, \ldots, u_m\}$, $\{v_1, v_2, \ldots, v_m\}$, $\{w_1, w_2, \ldots, w_m\}$ ...$\{z_1, z_2, \ldots, z_m\}$ respectively. These n-graphs are $n - 1$ connected by an edge between every pair of cycles, except the pair G_1, G_n to form a single graph.*

Then, $W(G_1 + G_2 + G_3 + \ldots + G_n)$
$= W(G_1) + W(G_2) + \ldots + W(G_n)$
$+W(G_1 +_2 G_2) + W(G_2 +_2 G_3)$
$+ \ldots + W(G_{n-1} +_2 G_n)+W(G_1 +_2 G_2 +_3 G_3)$
$+W(G_2 +_2 G_3 +_3 G_4)+\ldots+W(G_{n-2} +_2 G_{n-1} +_3 G_n) +\ldots + W(G_1 +_2 G_2 +_3 \ldots +_N G_n)$
$=n\, W(C_m)$

$$+\sum_{i=1}^{n-1}\sum_{j=2}^{n} W(G_i +_2 G_j)$$
$$+\sum_{i=1}^{n-2}\sum_{j=2}^{n-1}\sum_{k=3}^{n} W(G_i +_2 G_j +_3 G_k)$$

$$+\sum_{i=1}^{n-3}\sum_{j=2}^{n-2}\sum_{k=3}^{n-1}\sum_{l=4}^{n} W(G_i +_2 G_j +_3 G_k +_4 G_l) + \cdots$$

$$+\sum_{i_1=1}^{n-N}\sum_{i_2=2}^{n-(N-1)}\sum_{i_k=3}^{n-(N-2)}\cdots\sum_{i_k=N}^{n}W(G_{i_1}+_2G_{i_2}+_3\cdots+_NG_N)$$

where $n = 1, 2, ..., \infty$ *and* $i_1, i_2, ... i_k = 1, 2, ..., N$ *respectively.*

Proof:

Let $G_1, G_2, ..., G_n$ be n non-overlapping C_m graphs with m-cycle and n number of vertex sets, say,$\{u_1, u_2, u_3, u_4\}$, $\{v_1, v_2, v_3, v_4\}$, ..., $\{z_1, z_2, z_3, z_4\}$ respectively. n-bridge-connected graphs $W(G_1) +_2 W(G_2) +_3 W(G_3) +_4 ... +_n W(G_n)$ is obtained by joining sin- gle edge that has end point in $W(G_1) +_2 W(G_2) +_3 ... +_{n-1} W(G_{n-1})$ and G_n horizontally. Therefore, there exists a single edge whose end points lies in every pair of cycles (G_1, G_2, ...G_n) (except the end cycles) which are bridge connected systematically arranged in horizontal manner. Since,

$W(C_m) = \frac{1}{8}(m)^3$ *when m is even and*

$W(C_m) = \frac{(m-1)m(m+1)}{8}$, *when m is odd*, there will be n copies of C_m, $W(G_i) = W(C_m), m \geq 4$ Then,

$$W(G_1) = \sum_{i<j}^{n} d(u_i, u_j), W(G_2) = \sum_{i<j}^{n} d(v_i, v_j)$$

$$W(G_3) = \sum_{i<j}^{n} d(w_i, w_j) \ldots W(G_n) = \sum_{i<j}^{n} d(z_i, z_j)$$

Using the above example, we prove the following cases.

Case (i): When m is even, where $m \geq 4$

Let $m = 4$

From equation Eq. (4.1) to Eq. (4.4), we have

$W(G_1) + W(G_2) + ... + W(G_n)$

$=nW(C_4) + (n - 1)C_4(U_2) + (n - 2)C_4(U_3)$

$+ (n - 3)C_4(U_4)... + C_4(U_n)$ ....(4.5)

where,

$C_4(U_n)$ = 1[1(n -1 + 2t) + 2(n + 2t) + 1(n + 1 + 2t)]+ 2[1(n + 2t) + 2(n + 1 + 2t) + 1(n + 2 + 2t)]+1[1(n+1+2t) + 2(n +2+2t)+1(n+3 + 2t)]

..........(4.6)

Let $m = 6$

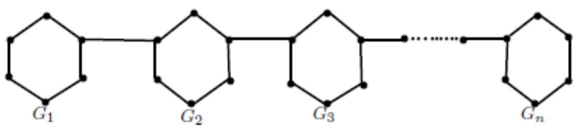

Figure.5: n-bridge connected C_6 graphs

From equation Eq. (4.1) to Eq. (4.4), we have

$W(G_1) + W(G_2) + ... + W(G_n)$

$=nW(C_6) + (n - 1)C_6(U_2) + (n - 2)C_6(U_3)$

$+ (n - 3)C_6(U_4)... + C_6(U_n)$ ....(4.7)

where,

$C_6(U_n) = 1[1(n-1 + 2t) + 2(n + 2t) + 2(n +1+ 2t) + 1(n + 2 + 2t)]+ 2[1(n + 2t) + 2(n + 1 + 2t) + 2(n + 2 + 2t) + 1(n + 3 + 2t)]+ 2[1(n + 1 + 2t) + 2(n + 2 + 2t) + 2(n + 3 + 2t) + 1(n + 4 + 2t)]+ 1[1(n + 2 + 2t) + 2(n + 3 + 2t) + 2(n + 4 + 2t) + 1(n + 5 + 2t)]$ ....(4.8)

Case (ii): When m is odd, where $m \geq 5$
Let m=5

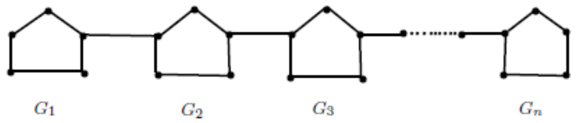

Figure.6: n-bridge connected C_5 graphs

From equation Eq. (4.1) to Eq. (4.4), we have
$W(G_1) + W(G_2) + \ldots + W(G_n)$
$= nW(C_5) + (n-1)C_5(U_2) + (n-2)C_5(U_3)$
$+ (n-3)C_5(U_4)\ldots + C_5(U_n)$ ....(4.9)
Where,
$C_5(U_n) = 1[1(n-1+2t) + 2(n+2t) + 2(n+1+2t)] + 2[1(n+2t) + 2(n+1+2t) + 2(n+2+2t)] + 2[1(n+1+2t) + 2(n+2+2t) + 2(n+3+2t)]$ ....(4.10)
Let m=7
From equation Eq. (4.1) to Eq. (4.4), we have
$W(G_1) + W(G_2) + \ldots + W(G_n)$
$= nW(C_7) + (n-1)C_7(U_2) + (n-2)C_7(U_3)$
$+ (n-3)C_7(U_4)\ldots + C_7(U_n)$ ....(4.11)
Where,

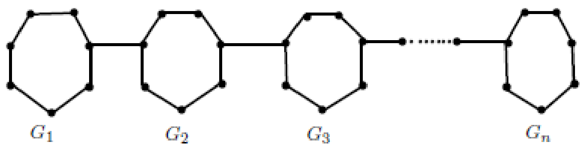

Figure.7: n-bridge connected C_7 graphs

$C_7(U_n) = 1[1(n-1+2t) + 2(n+2t) + 2(n+1+2t) + 2(n+2+2t)] + 2[1(n+2t) + 2(n+1+2t) + 2(n+2+2t) + 2(n+3+2t)] + 2[1(n+1+2t) + 2(n+2+2t) + 2(n+3+2t) + 2(n+4+2t)] + 2[1(n+2+2t) + 2(n+3+2t) + 2(n+4+2t) + 2(n+5+2t)]$............(4.12)
Generalizing from above Fig. 1 to Fig. 2, for m number of cycles, the n-bridge connected cycle graphs can be drawn as shown in Fig. 8.

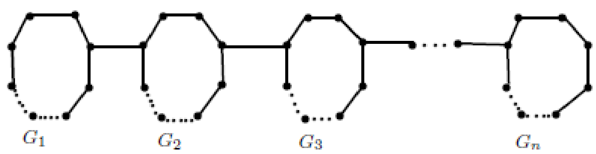

Figure.8: n-bridge connected C_m graphs

From equations Eq. (4.5) to Eq. (4.12) we observe that $C_m(U_n)$ cycles adheres the following sequence pattern.
When m is odd, $m \geq 5$. Let $k = 2, 3, 4, \ldots$.
m = 5 → (1 2 2)
m = 7 → (1 2 2 2)
m = 9 → (1 2 2 2 2)
.
.

.
m= 2k + 1 → (1 2 2 2 2 2….k times)
When m is even, $m \geq 4$. Let $k = 2, 3, 4, \ldots$.
m = 4 → (1 2 1)
m = 6 → (1 2 2 1)
m = 8 → (1 2 2 2 1)
.
.
.
m= 2k → (1 2 2 2 2 2….k-1 times 1)
Hence the general term is obtained using the above cases.

5. The Wiener index of n-bridge connected Star graphs S_m, m is odd or even:

Example 5.1:
5.1.1. The Wiener index of n-bridge connected S_3 graphs:
Let $G_1, G_2, G_3, \ldots, G_n$ be two non-overlapping S_3 graph with vertex sets $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$, ..., $\{z_1, z_2, z_3\}$ respectively. Two graphs $W(G_1) +_2 W(G_2)$ is obtained by joining $(n-1)$ degree vertex of G_1, say u_1 to $(n-1)$ degree vertex of G_2 by new edge. n-bridge-connected graphs $W(G_1) +_2 W(G_2) +_3 W(G_3) +_4 \ldots +_n W(G_n)$ is obtained by joining single edge that has end point with degree $(n-1)$ in $W(G_1) +_2 W(G_2) +_3 \ldots +_{n-1} W(G_{n-1})$ and G_n horizontally. Therefore there exists a single edge whose end points with degree $(n-1)$ lies in every pair of stars $(G_1, G_2, \ldots G_n)$ (except the end stars) which are bridge connected systematically arranged in horizontal manner. Since, $W(S_3) = (3-1)^2 = 4$, we have n copies of S_3,
Then $W(G_i) = 4, i = 1, 2, \ldots, n$

$$W(G_1) = \sum_{i<j}^{n} d(u_i, u_j), W(G_2) = \sum_{i<j}^{n} d(v_i, v_j)$$

$$W(G_3) = \sum_{i<j}^{n} d(w_i, w_j) \ldots W(G_n) = \sum_{i<j}^{n} d(z_i, z_j)$$

Since $W(G_1) +_2 W(G_2) +_3 \ldots +_{n-1} W(G_{n-1})$ and $W(G_n)$ are connected by an edge (y_1, z_1),

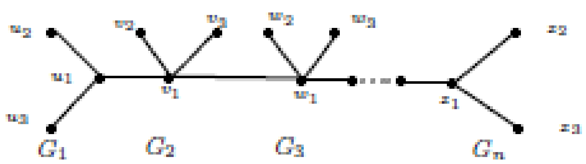

Figure.9: n-bridge connected S_3 graphs

$$W(G_1) +_2 W(G_2) +_3 \ldots +_n W(G_n)$$

$$= \sum_{i,j=1}^{3} d(z_i, u_j)$$

$= d(z_1, u_1) + \ldots + d(z_1, u_3)$
$+ d(z_2, u_1) + \ldots + d(z_2, u_3)$
$+ d(z_3, u_1) + \ldots + d(z_3, u_3)$
$=(n-1)+2(3-1)n+(3-1)^2(n+1)=S_3(U_n)$,(say)
$W(G_1)+W(G_2)+\ldots+W(G_n) = nW(S_3)+(n-1)U_2+(n-2)U_3+\ldots+U_n$ ....(5.1)
5.1.1. The Wiener index of n-bridge connected S_4 graphs:

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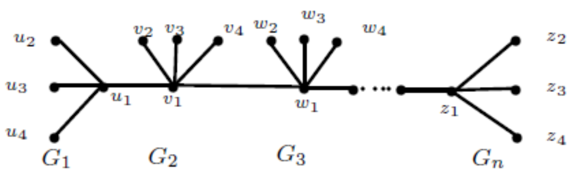


Figure.10: n-bridge connected S_4 graphs

From the above result, Since, $W(S_4) = (4-1)^2 = 9$, we have n copies of S_4,
Then $W(G_i) = 4, i = 1, 2, ..., n$
$W(G_1)+W(G_2)+...+W(G_n) =$
$nW(S_4)+(n-1)U_2+(n-2)U_3+...+U_n$ …….(5.2)
where,
$S_4(U_n) = (n-1) + 2(4-1)n + (4-1)^2(n+1)$

5.1.2. The Wiener index of n-bridge connected S_5 graphs:

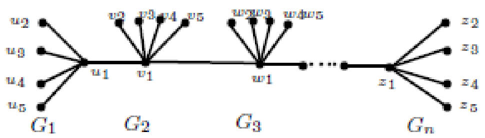


Figure.11: n-bridge connected S_5 graphs

From the above result, Since, $W(S_5) = (5-1)^2 = 25$, we have n copies of S_5,
Then $W(G_i) = 4, i = 1, 2, ..., n$
$W(G_1)+W(G_2)+...+W(G_n) =$
$nW(S_5)+(n-1)U_2+(n-2)U_3+...+U_n$ …….(5.2)
where,
$S_5(U_n) = (n-1) + 2(5-1)n + (5-1)^2(n+1)$

5.1.3. The Wiener index of n-bridge connected S_6 graphs:

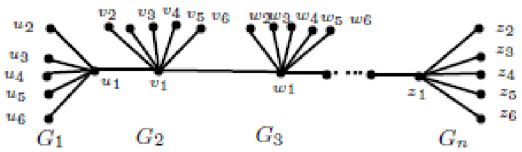


Figure.12: n-bridge connected S_6 graphs

From the above result, Since, $W(S_5) = (6-1)^2 = 36$, we have n copies of S_6,
Then $W(G_i) = 4, i = 1, 2, ..., n$
$W(G_1)+W(G_2)+...+W(G_n) =$
$nW(S_6)+(n-1)U_2+(n-2)U_3+...+U_n$ …….(5.2)
where,
$S_6(U_n) = (n-1) + 2(6-1)n + (6-1)^2(n+1)$

5.1.4. **Theorem 5.1:**
Let $G_1, G_2, ..., G_n$ be n distinct non-overlapping star graphs, S_m, $m \geq 3$ with vertex sets $\{u_1, u_2, ..., u_m\}$, $\{v_1, v_2, ..., v_m\}$, $\{w_1, w_2, ..., w_m\}$ ...,$\{z_1, z_2, ..., z_m\}$ respectively. These n-graphs are $n-1$ connected by an edge between every pair of stars, except the pair G_1, G_n to form a single graph. Then,
$W(G_1 + G_2 + G_3 + ... + G_n) = nW(S_m)$
$+ (n-1)S_m(U_2) + (n-2)S_m(U_3) + ... + S_m(U_n)$

where
$S_m(U_n) = (n-1) + 2(m-1)n + (m-1)^2(n+1)$

Proof:

Let $G_1, G_2, G_3..., G_n$ be n non-overlapping S_m graph with vertex sets$\{u_1, u_2, ..., u_n\}$ and $\{v_1, v_2, ..., v_n\}$, ..., $\{z_1, z_2, ..., z_n\}$ respectively. Two graphs $W(G_1) +_2 W(G_2)$ is obtained by joining $(n-1)$ degree vertex of G_1, say u_1 to $(n-1)$ degree vertex of G_2 by new edge. n-bridge-connected graphs $W(G_1) +_2 W(G_2)+_3W(G_3)+_4...+_nW(G_n)$ is obtained by joining single edge of star graph that has end point with degree $(n-1)$ in $W(G_1)+_2 W(G_2)+_3 ... +_{n-1} W(G_{n-1})$ and G_n horizontally. Therefore there exists a single edge whose end points with degree $(n-1)$ lies in every pair of stars $(G_1, G_2, ...G_n)$ (except the end stars) which are bridge connected systematically arranged in horizontal manner.

The Wiener index of these graphs is calculated by considering 2 at a time, then 3 at a time and proceeding in this way it is able to calculate the distances of all S_m graphs. Generalizing from above Fig. 9 to Fig. 12, for m number of stars, the n-bridge connected star graphs can be drawn as shown in Fig. 8.

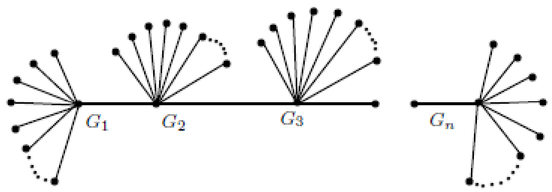


Figure.12: n-bridge connected S_m graphs

From the above example, we get, for $m \geq 3$, either m is odd or even,
$W(G_1) + W(G_2) = 2W(S_m) + S_m(U_2)$
$W(G_1) + W(G_2) + W(G_3) =$
$3W(S_m) + 2S_m(U_2) + S_m(U_3)$
$W(G_1) + ... + W(G_4) =$
$4W(S_m) + 3S_m(U_2) + 2S_m(U_3) + S_m(U_4)$
.
.
.
$W(G_1)+...+W(G_n) = nW(S_m)+(n-1)S_m(U_2) +(n-2)S_m(U_3)+...+S_m(U_n)$
where
$S_m(U_n) = (n-1) + 2(m-1)n + (m-1)^2(n+1)$
Hence the general term can be obtained using the above theorem,
That is, $W(G_1 + G_2 + G_3 + ... + G_n)$
$= W(G_1) + W(G_2) + ... + W(G_n)$
$+W(G_1 +_2 G_2) + W(G_2 +_2 G_3) + ... +$
$W(G_{n-1} +_2 G_n) +W(G_1 +_2 G_2 +_3 G_3)+$
$W(G_2 +_2 G_3 +_3 G_4)+...+$
$W(G_{n-2} +_2 G_{n-1} +_3 G_n)$
$+..... + W(G_1 +_2 G_2 +_3 .... +_N G_n)$
$=$n $W(C_m)$ +

$$+\sum_{i=1}^{n-1}\sum_{j=2}^{n} W(G_i +_2 G_j)$$

$$+\sum_{i=1}^{n-2}\sum_{j=2}^{n-1}\sum_{k=3}^{n} W(G_i +_2 G_j +_3 G_k)$$

$$+\sum_{i=1}^{n-3}\sum_{j=2}^{n-2}\sum_{k=3}^{n-1}\sum_{l=4}^{n} W\left(G_i +_2 G_j +_3 G_k +_4 G_l\right) + \cdots$$

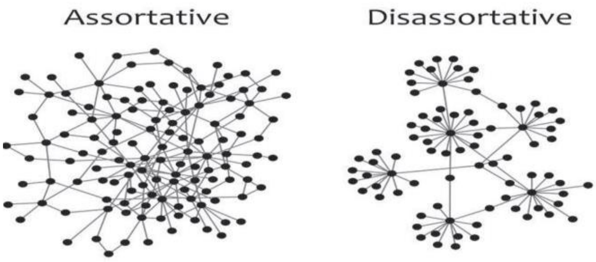
$$+\sum_{i_1=1}^{n-N}\sum_{i_2=2}^{n-(N-1)}\sum_{i_k=3}^{n-(N-2)} \cdots \sum_{i_k=N}^{n} W\left(G_{i_1} +_2 G_{i_2} +_3 \dots +_N G_N\right)$$

where $n = 1, 2, \dots, \infty$ and $i_1, i_2, \dots i_k = 1, 2, \dots, N$ respectively.

6. Application:

The human brain comprises 86 billion neurons connected through 150 trillion synapses that allow neurons to transmit electrical or chemical signals to other neurons [12]. The study of brain network using graph theory has been introduced by Sporns et al., [14] .

Graph theory can be related to model the communications between elements (nodes) of a network. In view of this a star graph resembles neural structure which can be applied to effective connectivity between every pair of nodes however long the neural structure may be. To access the topological pattern of these neural structure, wiener index plays a vital role in studying brain network. The complex structure of human brain can be studied with the help of highly connected nodes. To study the integration and segregation of network, in which there is a large distance between nodes, wiener index helps to reach any node from any other nodes irrespective of large length. Assortativity quantifies network resilience against random or deliberate damages in the main components, which is one of the most significant issues in network science [11]. The assortativity index measures the extent to which a network can resist failures in its main components (i.e., its vertices and edges). Notably, communication between hubs in assortative networks leads to covering each other's activities, but the performance in disassortative networks will drop sharply due to the presence of vulnerable hubs [5].



Courtesy: Frontiers in chemistry (Farahani et al.)

The cluster of star graphs taken in the above sections can be compared to the disassortative networks. Human intelligence can be related to brain imaging studies by analysing structure and functions of nodes when it is spatially distributed and shorter path length and this can be possible by finding the wiener index between these nodes. As the intelligent quotient is positively correlated with nodal properties and their distances, one can able to analyse the signals passing between these nodes and the general intelligence can be analysed using the n-bridge connected star graphs.

7. Conclusion:

In this paper Wiener index of n-bridge connected Cycle graph has been generalised for C_m graphs which has enormous applications in finding the distance between molecules of similar structure taken in this paper. Similarly, Wiener index of n-bridge connected Star graph has been generalised for S_m graphs which have extended applications in neural sciences.

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